Lecture 20: Physical States, Predictions & Measurements

Compatibility & Labelling of Quantum States

Previously: use eigenkets as basis kets – carries the optimal information about the associated observable only. Require a systematic way to label a quantum state for general optimum information.

Use the concept of *compatible observables* – two observables $\hat{A}, \hat{B}$ are compatible if

$$\left[\hat{A}, \hat{B} \right] = 0 \qquad .$$

If $\left[\hat{A}, \hat{B} \right] \neq 0$, the observables are said to be incompatible.

Theorem: Given compatible observables $\hat{A}, \hat{B}$ and that no eigenvalue of $\hat{A}$ is degenerate, then $\hat{B}$ has a diagonal matrix representation with respect to the basis eigenkets of $\hat{A}$. (Note: degenerate eigenvalues ≡ same eigenvalues that correspond to different eigenkets.)

Proof: Because of $\hat{B}$ is compatible with $\hat{A}$, then

$$\begin{aligned} 0 = \left\langle a_j \left| \left[\hat{A}, \hat{B} \right] \right| a_i \right\rangle &= \left\langle a_j \left| \left(\hat{A}\hat{B} - \hat{B}\hat{A} \right) \right| a_i \right\rangle \\ &= \left(a_j - a_i \right) \left\langle a_j \left| \hat{B} \right| a_i \right\rangle \qquad . \end{aligned}$$

For $j \neq i$, have assumed $a_j \neq a_i$

$$\Rightarrow \qquad \left\langle a_j \left| \hat{B} \right| a_i \right\rangle = 0 \quad , \qquad (j \neq i) \quad .$$

For $j = i$

$$a_j - a_i = 0 \qquad ; \qquad (j = i) \quad ,$$

but $\left\langle a_i \left| \hat{B} \right| a_i \right\rangle$ is not necessarily zero. One can now write

$$\left\langle a_i \left| \hat{B} \right| a_j \right\rangle = \delta_{ij} \left\langle a_i \left| \hat{B} \right| a_i \right\rangle \quad .$$

Matrix representation of $\hat{B}$ is diagonal:

$$\hat{B} = \begin{pmatrix} \langle a_1 | \hat{B} | a_1 \rangle & 0 & \cdots & 0 \\ 0 & \langle a_2 | \hat{B} | a_2 \rangle & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle a_n | \hat{B} | a_n \rangle \end{pmatrix} .$$

Other observation: the action of $\hat{B}$ of ket eigen of $\hat{A}$

$$\begin{aligned} \hat{B} | a_i \rangle &= \sum_{j=1}^{n} \sum_{k=1}^{n} | a_k \rangle \langle a_k | \hat{B} | a_j \rangle \langle a_j | a_i \rangle \\ &= \left(\langle a_i | \hat{B} | a_i \rangle \right) | a_i \rangle \equiv b_i | a_i \rangle \qquad ; \end{aligned}$$

i.e. ket $| a_i \rangle$ is also an eigenket of the compatible operator $\hat{B}$. Such kets are called *simultaneous eigenkets* for $\hat{A}$ and $\hat{B}$ which can be written as

$$| a_i \rangle \equiv | a_i , b_i \rangle \qquad \Rightarrow \qquad \begin{aligned} \hat{A} | a_i , b_i \rangle &= a_i | a_i , b_i \rangle \quad ; \\ \hat{B} | a_i , b_i \rangle &= b_i | a_i , b_i \rangle \quad . \end{aligned}$$

Generally, label the states completely with the eigenvalues of the maximal set of compatible observables

$$0 = \left[\hat{A}, \hat{B} \right] = \left[\hat{B}, \hat{C} \right] = \left[\hat{C}, \hat{A} \right] = \cdots$$

$$\Rightarrow \qquad \text{quantum states} \equiv \left| a_{i_1} , b_{i_2} , c_{i_3} , \cdots \right\rangle = | K_i \rangle \qquad ,$$

where

$$\langle K_i | K_j \rangle = \delta_{K_i K_j} = \delta_{i_1 j_1} \delta_{i_2 j_2} \delta_{i_3 j_3} \cdots \quad ;$$

$$\sum_{K_j} | K_j \rangle \langle K_j | = \sum_{j_1} \sum_{j_2} \sum_{j_3} \cdots | a_{j_1} , b_{j_2} , c_{j_3} , \cdots \rangle \langle a_{j_1} , b_{j_2} , c_{j_3} , \cdots | = \hat{1} \quad .$$

Applications: solve the case of labelling quantum states in the case where some observables have degenerate eigenvalues.

Example: Free particle state can be specified by the energy eigenvalue $E \ = \ p^2/2m$ but this label does not differentiate the state of the particle with momentum $+p$ or $-p$.

The solution is to use both eigenvalues for energy and momentum:

$$\left| E = p^2/2m, \ +p \right\rangle \qquad \text{atau} \qquad \left| E = p^2/2m, \ -p \right\rangle \quad .$$

Note: the value of E is automatically once p is known but not the other way round.

Probabilities, Predictions & Measurements

In relating quantum states with physical predictions in quantum mechanics, we must use the normalization condition on all kets:

$$\|\psi\|^2 \ \equiv \ \sqrt{\langle \psi | \psi \rangle} \ = \ 1 \qquad .$$

The required condition for probability interpretation:

$$\begin{aligned} \langle \psi | \psi \rangle \ &= \ \langle \psi | \sum_{i=1}^{n} | a_i \rangle \langle a_i | | \psi \rangle = \sum_{i=1}^{n} \langle \psi | a_i \rangle \langle a_i | \psi \rangle \\ &= \ \sum_{i=1}^{n} \left| \langle a_i | \psi \rangle \right|^2 \ = \ 1 \ \equiv \ \sum_{i=1}^{n} P(a_i) \qquad , \end{aligned}$$

where we have identified

$$\boxed{P(a_i) \ = \ \left| \langle a_i | \psi \rangle \right|^2 \ = \ \begin{array}{c} \text{probability of getting measured value } a_i \text{ for } \hat{A} \\ \text{in the system with the state } | \psi \rangle \end{array}}$$

Note:

- If observable $\hat{A}$ is measured on the system with state $|\psi\rangle$ that is not an eigenstate of $\hat{A}$, the produced measured value is *random* from amongst the possible eigenvalues (whose components exist in $|\psi\rangle$).
- If $|\psi\rangle \equiv |a_j\rangle$ is itself an eigenstate of $\hat{A}$, then on measuring $\hat{A}$ the results would surely gives the value a_j (unit probability).
- The quantity $\langle a_i | \psi \rangle$ is the expansion coefficient in

  $$|\psi\rangle = \sum_{i=1}^{n} |a_i\rangle\langle a_i|\psi\rangle$$

  and it is called the *probability amplitude*.

Even though the measurement results of $\hat{A}$ cannot be determined for a general state, one could always find the mean value:

$$\langle A \rangle = \sum_{i=1}^{n} a_i \, P(a_i) \quad .$$

Use the probabilistic expression of quantum theory ⇒ *expectation value:*

$$\langle \hat{A} \rangle_\psi = \sum_{i=1}^{n} a_i \left| \langle a_i | \psi \rangle \right|^2 = \sum_{i=1}^{n} \langle \psi | a_i \rangle a_i \langle a_i | \psi \rangle$$

$$= \sum_{i=1}^{n} \langle \psi | \hat{A} | a_i \rangle \langle a_i | \psi \rangle = \langle \psi | \hat{A} | \psi \rangle \quad .$$

This form can be generalised to get the expectation value of $\hat{O}$ in general:

$$\boxed{\langle \hat{O} \rangle_\psi = \langle \psi | \hat{O} | \psi \rangle}$$

Example: Particle in one-dimensional box $(-a, a)$ has the state

$$|\psi\rangle = \frac{1}{2} |E_2\rangle + \frac{\sqrt{3}}{2} |E_3\rangle$$

where $\left| E_j \right\rangle$ = energy eigenstate with the eigenvalue $E_j = \dfrac{\hbar^2 \pi^2 j^2}{8ma^2}$.

Expectation value of particle's energy is

$$\left\langle \psi \right| \hat{H} \left| \psi \right\rangle = \frac{1}{4} \left(\left\langle E_2 \right| + \left\langle E_3 \right| \sqrt{3} \right) \hat{H} \left(\left| E_2 \right\rangle + \sqrt{3} \left| E_3 \right\rangle \right)$$

$$= \frac{1}{4} \left(\left\langle E_2 \right| + \left\langle E_3 \right| \sqrt{3} \right) \left(E_2 \left| E_2 \right\rangle + \sqrt{3}\, E_3 \left| E_3 \right\rangle \right)$$

$$= \frac{1}{4} \left(E_2 \left\langle E_2 \middle| E_2 \right\rangle + \sqrt{3}\, E_2 \left\langle E_3 \middle| E_2 \right\rangle + \sqrt{3}\, E_3 \left\langle E_2 \middle| E_3 \right\rangle + 3\, E_3 \left\langle E_3 \middle| E_3 \right\rangle \right)$$

$$= \frac{1}{4} \left(E_2 + 0 + 0 + 3E_3 \right) = \frac{13 \hbar^2 \pi^2}{32 ma^2} \quad .$$

Energy E_i is measured with the probability:

$$P(E_2) = \left| \frac{1}{2} \right|^2 = \frac{1}{4} \quad ; \qquad P(E_3) = \left| \frac{\sqrt{3}}{2} \right|^2 = \frac{3}{4} \quad ; \qquad P(E_{i \neq 2,3}) = 0 \quad .$$

Note:

- When the measurement of $\hat{A}$ is made on the state $\left| \psi \right\rangle$ which produces the value a_j (with the probability $\left| \left\langle a_j \middle| \psi \right\rangle \right|^2$), then the state immediately after the measurement changes into the eigenstate $\left| a_j \right\rangle$. If subsequently $\hat{A}$ is measured once again, it will certainly reproduces the measured value a_j.

$$\left| \psi \right\rangle \xrightarrow{\text{measured } A = a_j} \left| a_j \right\rangle \xrightarrow{\text{measurement } A = a_j \text{ with certainty}} \left| a_j \right\rangle$$

- If $\left| \psi \right\rangle$ is measured consecutively for compatible observables $\hat{A}$ and $\hat{B}$, whose state $\left| a_j \right\rangle$ is simultaneous eigenstate of $\hat{B}$ with eigenvalue b_j and no a_j is degenerate, then measurement of $\hat{B}$ has value b_j with certainty. If $\hat{A}$ is measured once again, it will reproduce the earlier result of a_j.

$$\left|\psi\right\rangle \xrightarrow{\text{measured } A=a_j} \left|a_j, b_j\right\rangle \xrightarrow{\text{measurement } B=b_j \text{ with certainty}} \left|a_j, b_j\right\rangle$$

$$\xrightarrow{\text{measurement } A=a_j \text{ with certainty}} \left|a_j, b_j\right\rangle$$

- If the case of measurements of compatible $\hat{A}$ and $\hat{B}$ above has degenerate eigenvalues $a_1 = a_2 = \cdots = a_r \equiv a$, then the second measurement of $\hat{B}$ will not be giving measured values with certainty. Nevertheless the third measurement $\hat{A}$ reproduces the degenerate a.

$$\left|\psi\right\rangle \xrightarrow{\text{measured } A=a} \sum_{i=1}^{r}\left|a, b_i\right\rangle \xrightarrow{\text{measured } B=b_j \text{ with probability } \sum_{i=1}^{r}\left\langle b_j \middle| a, b_i\right\rangle} \left|a, b_j\right\rangle$$

$$\xrightarrow{\text{measurement } A=a \text{ with certainty}} \left|a, b_j\right\rangle$$

Review Questions:

20.1. State in a different way how two observables are compatible or not.

20.2. Given three mutually compatible observables $\hat{A}, \hat{B}, \hat{C}$. Construct their simultaneous eigenkets and form their completeness and orthonormal conditions.

20.3. Give examples of compatible observables and their simultaneous eigenstates.