Lecture 8: Issues Concerning Dirac Delta Function

Completeness of Hilbert Space

Example of incomplete vector space: Consider the vector space $C(a,b)$ of continuous functions over interval (a,b). Construct a sequence $\{f_k(x)\}$ defined on the interval $[-1,+1]$:

$$f_k(x) = \begin{cases} 1 & \text{if } k^{-1} \le x \le 1 \\ \frac{1}{2}(kx+1) & \text{if } -k^{-1} \le x \le k^{-1} \\ 0 & \text{if } -1 \le x \le -k^{-1} \end{cases} .$$

Scalar product: $$(f,g) = \int_{-1}^{1} f^*(x)\, g(x)\, dx \quad ; \qquad \| f \|^2 = (f,f)$$

$$\| f_i - f_j \| = \int_{-1}^{1} \left| f_i - f_j \right|^2 dx \xrightarrow[i,j\to\infty]{} 0 .$$

Therefore, $\{f_k(x)\}$ = Cauchy sequence. Limit of sequence?

$$f_j \to f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } -1 < x < 0 \end{cases} \notin C(-1,1)$$

$\Rightarrow$ requires completion of the set of continuous functions with discontinous ones.

Set of square-integrable functions for Hilbert space

- complete – includes discontinuous functions (example: square wave)
- infinite dimensional vector space – has **countably** infinite number of basis functions

How about aperiodic functions – requires superposition with continuous *k* values – uncountable number of basis functions? What are the modifications or inclusions necessary?

Dirac Delta Function

Square integrable functions – normalizable – requirement for probability interpretation.

Consider function of plane waves: $\psi_k(x) \; = \; \frac{1}{\sqrt{2\pi}} \, e^{ikx}$

Normalization of plane waves?

$$\int_{-\infty}^{\infty} \psi_k^*(x)\,\psi_k(x)\,dx \; = \; \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; = \; \frac{1}{2\pi}\,[\,x\,]_{-\infty}^{+\infty} \; = \; ???$$

Infinite extent of plane waves not qualified?
Extend Hilbert space? – require a new kind of mathematical object – *Dirac delta function/distribution*

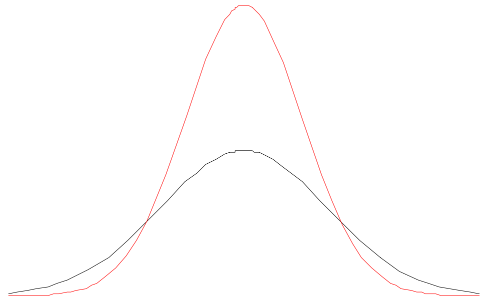
Consider a Gaussian function with peak at $x = x_0$ and width 2σ:

$$\psi_\sigma(x - x_0) \; = \; \frac{1}{2\sigma\sqrt{\pi}} \, \exp\left\{ -\frac{(x-x_0)^2}{4\sigma^2} \right\}$$

and satisfying

$$\int_{-\infty}^{\infty} \psi_\sigma(x - x_0)\,dx \; = \; 1 \quad . \qquad (*)$$

The peak's height of ψ_σ is inversely proportional with its width: if width 2σ gets narrower, amplitude $(2\sigma\sqrt{\pi})^{-1}$ grows higher.



Area under the graph remains the same – equation (*) holds for all σ.

Take the limit $\sigma \to 0$ (zero width) $\Rightarrow$ peak's height $\to \infty$; producing *Dirac delta function*

$$\delta(x-x_0) = \underset{\sigma \to 0}{\text{had}} \; \frac{1}{2\sigma\sqrt{\pi}} \exp\left\{ -\frac{(x-x_0)^2}{4\sigma^2} \right\} \quad ;$$

i.e.

$$\boxed{\delta(x-x_0) = \begin{cases} 0 & \text{bagi } x \neq x_0 \\ \infty & \text{bagi } x = x_0 \end{cases}}$$

The property (*) still holds:

$$\boxed{\int_{-\infty}^{\infty} \delta(x-x_0)\, dx = 1}$$

Other properties:

$$\boxed{\int_{-\infty}^{\infty} \delta(x-x_0)\, g(x)\, dx = g(x_0)}$$

analogous to

$$\sum_n C_n \, \delta_{nm} = C_m \quad .$$

Derivative of Dirac delta:

$$\boxed{\int_{-\infty}^{\infty} \left[\frac{d}{dx} \delta(x-x') \right] g(x')\, dx' = \frac{dg(x)}{dx}}$$

Note: Dirac Delta – not really a function – can be given in different formulae:

$$\delta(x-x_0) = \lim_{\alpha \to \infty} \frac{1}{\pi} \frac{\sin \alpha(x-x_0)}{(x-x_0)} \quad ;$$

$$\delta(x-x_0) = \frac{d}{dx} \theta(x-x_0) \quad ,$$

$$\theta(x-x_0) = \begin{cases} 1 & \text{for } x > x_0 \\ 0 & \text{for } x < x_0 \end{cases} \quad .$$

Continuous Normalization of Plane Waves

Integral form of Dirac delta function – use Fourier pair:

$$\psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \underbrace{\int_{-\infty}^{\infty} dx'\, \psi(x')\, e^{-ikx'}}_{\sqrt{2\pi}\,\tilde{\psi}(k)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, \psi(x') \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} \quad .$$

Compare this with

$$\psi(x) = \int_{-\infty}^{\infty} dx'\, \psi(x')\, \delta(x'-x) \quad .$$

$$\boxed{\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')}}$$

If write plane waves as $\psi_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$, then

$$\int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \psi_{k'}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{i(k'-k)x} = \delta(k'-k) \qquad (\triangle)$$

analogous to orthonormal vector identity:

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

Note:

- ‘Normalization’ in $(\triangle)$ is called *continuous normalization* of plane waves – not the usual normalization.
- If further ‘summed’ over k, answer gives unity:

$$\int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dx\, \psi_k^*(x)\, \psi_{k'}(x) = \int_{-\infty}^{\infty} dk'\, \delta(k'-k) = 1 \ .$$

Box Normalization of Plane Waves

Usually assume physical system is not influenced by factors from outside some region (example: laboratory space).

Assume plane wave is limited to some box region:

$$\psi_k \;=\; \begin{cases} A e^{i(\underline{k}\cdot\underline{r} - \omega t)} & \text{for } \underline{r} \text{ in box region } V = L^3 \\ 0 & \text{for } \underline{r} \text{ outside box region } V = L^3 \end{cases} \quad .$$

Need to impose additional condition, for example

$$\psi(x+L, y, z) \;=\; \psi(x, y+L, z) \;=\; \psi(x, y, z+L) \;=\; \psi(x, y, z)$$

– will involve quantization of wavenumbers (later).

Normalization process:

$$\int_{\substack{\text{all}\\ \text{space}}} d^3r \; |\psi|^2 \;=\; A^2 \int_{\substack{\text{box}\\ V = L^3}} d^3r \;=\; A^2 L^3 \equiv 1 \quad .$$

$\Rightarrow$ normalization constant $A \;=\; L^{-3/2} = V^{-1/2}$

Normalized function of plane waves:

$$\psi_k(\underline{r}, t) \;=\; \frac{1}{\sqrt{V}}\; e^{i(\underline{k}\cdot\underline{r} \,-\, \omega t)} \quad .$$

One-dimensional case:

$$\psi_k(x) \;=\; \frac{1}{\sqrt{L}}\; e^{ikx} \quad .$$