# Lecture 9: Quantum State Postulate

## *Momentum Probability Function*

Previously: wavefunction $\psi(x)$ as probability amplitude for particle's position.

Q: Probabilistic information about momentum? Energy? …

Use deBroglie relation $p = \hbar k$ and Fourier pair $\tilde{\psi}(k)$ for $\psi(x)$.
Define momentum probability function with normalized $\tilde{\psi}(k)$:

$$P(k) = \tilde{\psi}^*(k)\,\tilde{\psi}(k)$$

where

$P(k)\,dk$ = probability of particle's momentum between $\hbar k$ and $\hbar(k+dk)$ .

Probability of particle's momentum between $p_1$ and $p_2$:

$$P(p_1 \le p \le p_2) = \int_{p_1/\hbar}^{p_2/\hbar} \left|\tilde{\psi}(k)\right|^2 dk$$

*Example:* Given a particle is represented by the pulse

$$\psi(x) = \begin{cases} 2^{-1/2} & \text{if } -1 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

has the Fourier pair

$$\tilde{\psi}(k) = \sqrt{\frac{1}{\pi}}\,\frac{\sin(k)}{k} \quad .$$

The probability of particle's momentum between $-h$ and $h$ is

$$P(-h \le p \le h) = \frac{1}{\pi}\int_{-2\pi}^{2\pi} \frac{\sin^2 k}{k^2}\,dk = 0.949939 \quad .$$

Note:

- Interpretation of $\tilde{\psi}(k)$ as probability amplitude requires normalised property.
- Should there be an extra normalization constant for $\tilde{\psi}(k)$?

*Proposition:* If $\psi(x)$ is normalized, then its Fourier pair $\tilde{\psi}(k)$ is also normalized and vice versa.

*Proof:* Assume $\psi(x)$ is normalized. Consider the corresponding momentum probability function and integrate it to get the total probability:

$$
\begin{aligned}
\int_{-\infty}^{\infty} P(k)\,dk &= \int_{-\infty}^{\infty} \tilde{\psi}^*(k)\,\tilde{\psi}(k)\,dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx\ \psi^*(x')\,\psi(x)\,e^{-ik(x-x')} \\
&= \lim_{\kappa\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx\ \psi^*(x')\,\psi(x) \int_{-\kappa}^{\kappa} dk\, e^{ik(x'-x)} \\
&= \lim_{\kappa\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx\ \psi^*(x')\,\psi(x)\ \frac{\sin\left[\kappa(x'-x)\right]}{(x'-x)} \\
&\simeq \lim_{\kappa\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} dx'\,\psi^*(x')\,\psi(x')\int_{-\infty}^{\infty} dx\ \frac{\sin\left[\kappa(x'-x)\right]}{(x'-x)} \\
&= \int_{-\infty}^{\infty} dx'\,\psi^*(x')\,\psi(x')\int_{-\infty}^{\infty} dx\ \ \delta(x'-x) \\
&= \int_{-\infty}^{\infty} dx'\,\psi^*(x')\,\psi(x') \;=\; 1
\end{aligned}
$$

Have used the identity: $\delta(x-x_0) = \lim_{\alpha\to\infty} \frac{1}{\pi}\,\frac{\sin\alpha(x-x_0)}{(x-x_0)}$.

*Exercise:* Prove the reverse.

***Quantum States***

Above discussion: probabilistic information of position and momentum of particle can be obtained from $\psi(x)$ (for momentum through Fourier pair).

Alternative: probabilistic information of position and momentum of particle can also be obtained from $\tilde{\psi}(k)$ (for position through Fourier pair).

In principle, either one of $\psi(x)$ or $\tilde{\psi}(k)$ is sufficient to carry probabilistic information of any dynamical physical quantity (as functions of position and momentum) – satisfying one of the conditions for the *quantum state* concept.

Quantum state – an abstract concept – has a concrete form with respect to a particular representation:

- *Position representation:* quantum state = $\psi(x)$;
- *Momentum representation:* quantum state = $\tilde{\psi}(k)$.

There exists other representations (later).

Comparison: Classical states – from real kinematic variables/quantities

Question: quantum states as complex functions – can they be given real form?

For example, wavefunction as function of position in polar form:

$$\psi(x) \;=\; R(x)\, e^{i\alpha(x)}$$

where $R(x), \alpha(x)$ are real functions, giving probability function

$$P(x) \;=\; R(x)\, e^{-i\alpha(x)}\, R(x)\, e^{i\alpha(x)} \;=\; \left(R(x)\right)^2 \quad .$$

Can we ignore the function $\alpha(x)$? How about the momentum probability function?

Know

$$P(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx \;\; \psi^*(x')\,\psi(x)\, e^{-ik(x'-x)}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx \;\; R(x')\, R(x)\, e^{-i[k(x'-x)+\alpha(x')-\alpha(x)]} \quad .$$

The dependence on $\alpha(x)$ or $\alpha(x')$ cannot be eliminated – quantum states must be represented by complex functions.

Special case: photon's wavefunction can be real because position aspects of propagating photons are not meaningful.

Quantum state postulate: In maximal information condition, a quantum state as the carrier of probabilistic information will be represented as a vector in a complex Hilbert space $\mathcal{H}$.

Hilbert space $\mathcal{H}$ = space of quantum states.

Problem:

- Probabilistic concepts usually involve an ensemble of identical systems.
- What kind of probabilistic information is carried by an individual quantum system?